The Remainder Theorem and Synthetic Substitution/Division

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Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$ be a polynomial of degree *n*.

If *P*(*x*) is divided by *x* − *b*, then

$$\frac{P(x)}{x-b} = Q(x) + \frac{R}{x-b},$$

where *Q*(*x*) is the quotient, a polynomial of degree *n* − 1, and *R* is the remainder, a constant. Multiplying both sides by *x* − *b* gives

$$P(x) = (x-b)Q(x) + R.$$

If the polynomial is evaluated at *x* = *b*, then

$P(b) = (b-b)Q(b) + R = 0 \cdot Q(b) + R = R$. So $R = P(b)$ and

$P(x) = (x-b)Q(x) + P(b)$. This is ***The Remainder Theorem***.

So, to find *P*(*b*) you can divide *P*(*x*) by *x* – *b* and the remainder will be *P*(*b*).

If *P*(*b*) = 0, then *x* − *b* is a *factor* of *P*(*x*).

Of course you can always directly substitute *b* for *x* in the polynomial, like this:

$P(b) = a_n b^n + a_{n-1}b^{n-1} + ... + a_1 b + a_0$, and using some unusual factoring,

$P(b) = (((a_n)b + a_{n-1})b + ... + a_1)b + a_0$, which is called *The Nested Form*.

The Nested Form represents a procedure that can be tabularized:

	x^n	x^{n-1}	...	x^1	1
b	a_n	a_{n-1}	...	a_1	a_0
(+)		$(a_n)b$	...	$(((a_n)b + a_{n-1})b + ... + a_2)b$	$((((a_n)b + a_{n-1})b + ... + a_2)b + a_1)b$
	a_n	$(a_n)b + a_{n-1}$	...	$(((a_n)b + a_{n-1})b + ... + a_2)b + a_1$	$P(b)$

This is known as *Synthetic Substitution*. For a concrete example, let $f(x) = 2x^3 - 5x^2 + 7$, then *f* (3) is found using Synthetic Substitution as follows in the figure at the left.

	x^3	x^2	x^1	1	
3	2	−5	0	7	
(+)		6	3	9	
	2	1	3	16	$= f(3)$

		x^3	x^2	x^1	1	
	3	2	−5	0	7	← $f(x)$
	(+)		6	3	9	← $bQ(x)$
$xQ(x)$ →		2	1	3	16	$= f(3)$

The Remainder Theorem can be used to understand what else is being shown in *Synthetic Substitution*. If $P(x) = (x-b)Q(x) + P(b)$, then, with a little algebraic manipulation, $P(x) + bQ(x) = xQ(x) + P(b)$; which shows that *Synthetic Substitution* also finds the quotient *Q*(*x*) of a division of *P*(*x*) by *x* − *b*, as annotated for our concrete example in the above figure on the right. So *Synthetic Substitution* is also known as *Synthetic Division*.